J. Fluid Mech. (1974), *vol.* 63, *part* 1, *pp.* 157–175
Printed in Great Britain

A mathematical model of mass transport in a long permeable tube with radial convection

By STEPHEN M. ROSS

Department of Mechanical Engineering, State University of New York, Buffalo

(Received 13 February 1973 and in revised form 12 July 1973)

Laplace transforms and regular double asymptotic expansions are used to solve the problem of ordinary chemical mass transport in a permeable tube, where there is small radial convection through the membrane wall and where the length-to-diameter ratio is large. The system is taken to be dilute and Newtonian and the solution is found to higher order in two small parameters. Results indicate that the exit concentration decreases markedly as the diameter, membrane permeability and tube length increase, and that changes in mass transport owing to variations in radial convection are much more significant than those due to the same order of magnitude changes in the resistance of the chemical solute to passage through the membrane (transmittance). In addition, the maximum effects of changes in the radial convection and transmittance are not at the membrane itself ($r = 1$), but rather roughly at radial values of 0·6 and 0, respectively.

1. Introduction

An analysis has been undertaken to provide a model for mass transport in laminar flow of a Newtonian fluid through a permeable tubular membrane, where there exists a small radial flux of fluid (ultra-filtration). Although the model is motivated by the study of blood flow in a hollow-fibre type of artificial kidney which consists of bundles of these tubes, the equations are solved in generality, so that other possible applications, such as in desalination and heat transfer, may be included in the solution. Only those findings pertinent to the artificial kidney, however, will be discussed in this paper.

The literature on both reverse osmosis in desalination and laminar heat transfer is closely related to the subject under consideration here. In reverse osmosis, the diffusion equation is solved for a boundary condition in which the flux of solute (salt) through the membrane is due solely to the solvent (water) flux. The fluid velocity through the membrane depends on both the hydrostatic and osmotic pressure difference, and the solute dragged through the membrane is proportional to the 'rejection coefficient'. According to Probstein's (1972) review of desalination, it appears that solutions are obtained for simplified systems in which the velocity field is given in terms of some known bulk ultra-filtration velocity, which is taken to be either constant along the membrane or dependent on the concentration (Brian 1966). Thus, the reverse-osmosis problem differs from the one

considered in this paper in several ways. In reverse osmosis, the flux of solute through the membrane arises solely from the mechanism of ultra-filtration, and the ultra-filtration velocity depends on the osmotic pressure, whereas osmotic pressure is a negligible effect in this analysis. In addition, the ultra-filtration velocity is not determined from the hydrodynamic equations as a function of distance along the membrane in the reverse-osmosis literature.

By and large, in most of the laminar flow heat-transfer literature, the diffusion equation is solved with wall boundary conditions of constant temperature or constant heat flux, neither of which correspond to the mass-transport problem under discussion in this paper. This classic 'Graetz' problem has been studied extensively (e.g. Kays 1966; Goldstein 1938); but the type of boundary condition that includes the convection of heat or mass through the membrane by a velocity dependent on the pressure difference has not been considered.

Grimsrud & Babb (1966) were the first to solve the problem of chemical diffusion in blood flowing through two infinite flat plates, following the theory of De Bye & Schenk (1953). Colton *et al.* (1971) solved the same problem by using asymptotic methods to determine higher eigenvalues, so that the entrance region (with respect to concentration) could be studied. Both Grimsrud and Colton assumed that the flow down the plates was strictly Poiseuillian, i.e. zero ultra-filtration.

Popovich *et al.* (1971) considered the effect of non-zero fluid velocity through the plates by modifying the boundary condition to include convective as well as diffusive transport, and by allowing two components of velocity. The velocity field used in their study was taken from the work of Berman (1953), who solved the Navier–Stokes equation asymptotically for small values of the ultra-filtration velocity, which was specified constant at the plate. Although this last assumption simplified the analysis to a large extent, it was not realistic, since the ultra-filtration velocity should vary with the pressure difference across the membrane to a first approximation, so that it should be determined from the hydrodynamic equations as a function of distance.

In this paper, we consider the mass transport in a Newtonian fluid flowing steadily through a small-diameter tube, in which the radial velocity through the membrane is not known *a priori*, but is proportional to the pressure difference. The tube length-to-diameter ratio is assumed large enough so that there are no end effects. The solution is found for the case in which the net flux of fluid through the tube wall is smaller than the average axial flux. This condition is clearly valid both when radial convection is 'small' in some sense, and when we confine the solution to a bounded region axially. Both conditions are approximated in the actual hollow-fibre dialyser.

The concentration (mass fraction) of solute is a given constant c_I at an arbitrary 'entrance' point designated by $z = 0$, and we solve for it as a function of space and the parameters of the system.

2. Field equations

The flow is steady and independent of angle in cylindrical co-ordinates and the solution is dilute so that the viscosity μ and total mass density ρ may be taken

as constants. The components of velocity are denoted by $\mathbf{u} = (u, w)$ in the radial r and axial z directions, respectively; P denotes pressure, D molecular diffusivity ($\sim 10^{-5}\,\mathrm{cm^2\,s^{-1}}$), a tube radius ($\sim 10^{-2}\,\mathrm{cm}$), W mean axial velocity ($\sim 1\,\mathrm{cm\,s^{-1}}$), $\hat{P}$ solute membrane permeability ($\sim 10^{-3}\,\mathrm{cm^2\,s^{-1}}$), and K hydrodynamic permeability ($\sim 10^{-11}\,\mathrm{cm}$). The actual geometric fibre length is $\sim 10\,\mathrm{cm}$. The basic data are taken to correspond roughly to the flow of one chemical solute, such as urea through a single fibre of a typical hollow-fibre hemodialyser; in this case, one manufactured by Cordis–Dow.

Non-dimensionalization. There are four independent dimensionless numbers that automatically arise from the scaling. The most obvious choice for scaling r and w is $r = ar^*$ and $w = Ww^*$, where the * denotes dimensionless quantities. However, there are two choices for the radial velocity scale: the mean ultra-filtration velocity U and the diffusion velocity D/a. Since

$$D/a \sim 10^{-3}\,\mathrm{cm\,s^{-1}} \quad \text{and} \quad U < 10^{-4}\,\mathrm{cm\,s^{-1}},$$

the radial diffusion speed dominates, and we choose to scale $u = Du^*/a$. Also, the axial length scale l is based on the ratio of axial velocity to radial velocity rather than the geometric length. Thus,

$$l/a = W/(D/a) \quad \text{or} \quad l = a^2W/D$$

(i.e. $l \sim 10\,\mathrm{cm}$) and so $z = a^2Wz^*/D$. The pressure is non-dimensionalized so that the gradient in the axial direction is the same order as the viscous forces ($P = \mu W^2P^*/D$).

Substitution of the above into the equations defining the model gives the following dimensionless constants:

$$\epsilon = \frac{D}{aW} = \frac{\text{radius}}{\text{axial length scale}} \sim 10^{-3},$$

$$\delta = \frac{KaW^2}{D^2} \sim \text{ultra-filtration number} \sim 10^{-3},$$

$$R = \rho\frac{Wa}{\mu} \sim \text{Reynolds number} \sim 1,$$

$$S = \frac{\hat{P}a}{D} \sim \text{Sherwood number} \sim 1.$$

In the related heat-transfer problem (Kays 1966, ch. 8), the dimensionless numbers S and ϵ correspond to the Nusselt and Prandtl numbers (omitting specific heat and density): $S \sim N_u$ and $\epsilon \sim (RP_r)^{-1}$. Thus, $l \sim aRP_r$, so that the scaling in this problem corresponds to that in the heat-transfer problem.

The dimensionless equations of motion and total mass conservation for a Newtonian fluid in cylindrical co-ordinates are (omitting *)

$$\epsilon^3R\mathbf{u}.\nabla u = -P_{,r} + \epsilon^2\left\{\frac{1}{r}(ru_{,r})_{,r} - \frac{u}{r^2}\right\} + \epsilon^4u_{,zz}, \tag{2.1}$$

$$\epsilon R\mathbf{u}.\nabla w = -P_{,z} + \frac{1}{r}(rw_{,r})_{,r} + \epsilon^2w_{,zz}, \tag{2.2}$$

$$\frac{1}{r}(ru)_{,r} + w_{,z} = 0, \tag{2.3}$$

$$\mathbf{u}.\nabla(\ ) \equiv u(\ )_{,r} + w(\ )_{,z}. \tag{2.4}$$

The dimensional equation expressing conservation of mass of solute in steady state flow (Aris 1962) is

$$\rho \mathbf{u} . \nabla c + \nabla . \mathbf{j} = 0, \tag{2.5}$$

where c is the mass fraction of solute ρ_α/ρ, $\mathbf{j} \equiv \rho_\alpha(\mathbf{u}_\alpha - \mathbf{u})$ is the flux of solute relative to the mass average velocity $\mathbf{u}$, ρ_α is the mass density of solute, and $\mathbf{u}_\alpha$ is the solute velocity. For a dilute solution in which thermal, pressure and forced diffusion are neglected, the constitutive relation between $\mathbf{j}$ and concentration is taken to be Fickian (Bird *et al.* 1960):

$$\mathbf{j} = -\rho D \nabla c, \tag{2.6}$$

so that (2.5) and (2.6) give the dimensionless diffusion equation

$$\mathbf{u} . \nabla c = \frac{1}{r}(r c_{,r})_{,r} + \epsilon^2 c_{,zz}. \tag{2.7}$$

Since ϵ and δ are small parameters of the same order of magnitude, it is necessary to consider both effects in the solution to follow. Clearly, neglecting terms of $O(\epsilon^2)$ in (2.7) means that axial diffusion is small compared to radial diffusion; however, first-order ϵ terms enter (2.7) in the velocity field. These $O(\epsilon)$ terms must be included, if a solution is sought correct to $O(\delta)$, since both dimensionless numbers are of the same order of magnitude.

In addition to the above observations, we may expect this problem to be singular near the z boundaries, because highest-order derivatives are multiplied by powers of ϵ in (2.1), (2.2) and (2.7). This fact will be discussed in § 4.

3. Boundary conditions

At the porous wall, the non-slip condition is assumed valid axially, and the radial velocity is taken to be proportional to the pressure difference across the wall (figure 1). These are the simplest physically reasonable boundary conditions for slightly hydrodynamically permeable membranes. Osmotic pressure effects are neglected, since they can be eliminated artificially by suitable molecular additions to the outer dialysate fluid space. In dimensionless form:

$$u = \delta(P - P_0) \quad \text{on} \quad r = 1 \quad \text{for all } z, \tag{3.1}$$

$$w = 0 \quad \text{on} \quad r = 1 \quad \text{for all } z, \tag{3.2}$$

where δ is the 'ultra-filtration' number, and P_0 is the constant outer pressure assumed to approximate the turbulent dialysate flow condition. Equation (3.1) differs from the boundary condition in reverse osmosis, since the osmotic pressures are neglected and u is independent of the concentration in this problem.

The boundary conditions are deliberately posed, so that there is a basic Poiseuille flow, and no end conditions are imposed on the velocity field. Superimposed on this main flow, however, is the effect of a small, z-dependent, radial convection through the tube walls. In addition, the pressure and pressure gradient must be supplied at the entrance $z = 0$ (i.e. $P = P_I$ and $P_{,z} = G$), where P_I and G are known in terms of ϵ and δ.

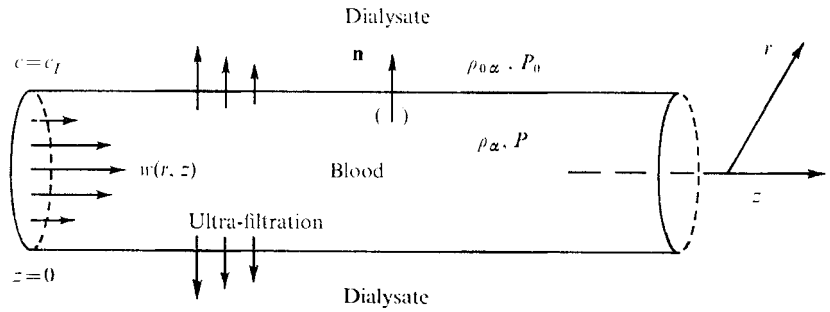

FIGURE 1. Diagram of a single fibre of radius a with concentration c_I at $z = 0$. The axial velocity is denoted by $w(r, z)$ and the mass densities of solute α inside and outside the tube are ρ_α and $\rho_{0\alpha}$, respectively. The dialysate pressure P_0 is a constant in space, and the concentration c of any solute α is defined as ρ_α/ρ, where ρ is the total density. The dialysate concentration is negligible relative to the blood concentration as far as diffusion across the membrane is concerned.

According to Kays (1966), Langhaar's work on entry length in a tube indicates that fully developed laminar flow is reached when $z/a \cong \frac{1}{10}R$, so that in the problem considered in this paper the entry length is $\sim 10^{-3}$ cm. Thus, entrance conditions are taken to be fully developed flow (in the sense of no entrance effect) and constant concentration:

$$c = c_I \quad \text{at} \quad z = 0 \quad \text{for all } r. \tag{3.3}$$

The mass flux of chemical solute through the wall depends on the amount carried through by both radial convection and ordinary diffusion. Following Popovich (1971), the boundary condition is taken to be the synthesis of a diffusive flux depending on the membrane permeability $\hat{P}$ (as in Colton's (1971) hemodialysis analysis) and a convective flux depending on the ultra-filtration (as in Probstein's (1972) or Brian's (1966) reverse-osmosis analyses). The *solute* boundary condition, then, reduces to that in the desalination literature when $\hat{P} = 0$, and to that in hemodialysis literature when $u = 0$.

In effect, it is necessary to postulate a constitutive equation for the total mass flux of solute through the membrane wall as a function of the radial velocity, the density of solute ρ_α and the density of solute in the outer dialysate space $\rho_{0\alpha}$. The total mass flux of solute per unit area through the tube wall is assumed to be a linear function of $\mathbf{u}$, ρ_α and $\rho_{0\alpha}$ (figure 1). In dimensional terms,

$$\rho_\alpha \mathbf{u}_\alpha . \mathbf{n} = \hat{k}\rho_\alpha \mathbf{u} . \mathbf{n} + \hat{P}(\rho_\alpha - \rho_{0\alpha}) \quad \text{on} \quad r = a, \quad z > 0, \tag{3.4}$$

where $\mathbf{n}$ is the outward normal. The term $\hat{k}\rho_\alpha \mathbf{u} . \mathbf{n}$ is the flux through the wall due to radial convection of the bulk fluid, and $\hat{k}$ is a retarding factor called the 'transmittance', which, in the absence of ordinary diffusion through the membrane ($\hat{P} = 0$), may be defined as the ratio of solute flux through the membrane to the solute flux convected to the membrane by the mass average velocity Brian 1966). (Therefore, $\hat{k}$ ranges from 0 to 1, and may be visualized via a membrane pore model as $\cong 1$ if the membrane pore size is very much greater than the solute molecular size, and $\cong 0$ if the pore size is very much less than

the molecular size. The sum of $\hat{k}$ and the 'rejection coefficient' in reverse osmosis (Liu 1971; Brian 1966) equals 1. The term $\hat{P}(\rho_\alpha - \rho_{0\alpha})$ represents ordinary Fickian molecular diffusion through the wall.

Using the definition of $\mathbf{j}$, which states that $\rho_\alpha \mathbf{u}_\alpha = \mathbf{j} + \rho_\alpha \mathbf{u}$, and dividing by ρ, we get

$$\{c\mathbf{u} - D\nabla c\}.\mathbf{n} = \hat{k}c\mathbf{u}.\mathbf{n} + \hat{P}(c - c_0) \quad \text{on} \quad r = a, \quad z > 0, \tag{3.5}$$

where $c_0 \equiv \rho_{0\alpha}/\rho$ is the outer fluid concentration. Neglecting c_0 relative to c and rearranging terms, the dimensionless form of this boundary condition may be written as

$$c_{,r} + \beta c = 0 \quad \text{on} \quad r = 1, \quad z > 0, \tag{3.6}$$

where

$$\beta = \beta(z) \equiv (\hat{k} - 1)\, u|_{r=1} + S, \tag{3.7}$$

and S is the Sherwood number.

Equation (3.6) reduces to Probstein's (1972) and Brian's (1966) boundary condition when $S = 0$, and to Colton's (1971) when $u = 0$. The assumption that c_0 is small relative to c approximates the condition in a hemodialyser, where the dialysate concentration enters at zero concentration and high flow rate.

Existence conditions in cylindrical co-ordinates where $r = 0$ may be singular are

$$\lim_{r \to 0} |\mathbf{u}| < \infty, \quad \lim_{r \to 0} |c| < \infty. \tag{3.8), (3.9}$$

4. Asymptotic expansion

The hydrodynamic and mass-transport problems defined by (2.1), (2.2), (2.3), (3.1), (3.2), (3.8) and (2.7), (3.3), (3.6), (3.9), respectively, are mathematically uncoupled, in the sense that we can solve for the velocity field first without considering the mass transport. This is, naturally, the result of the fact that the solution is dilute and viscosity is assumed concentration independent. The mass-transport equations are solved once the velocity field is determined.

Since ϵ and δ are two small parameters of the same order, we seek a solution correct to first order in the doubly asymptotic form

$$f = f^{(0,0)} + \epsilon f^{(1,0)} + \delta f^{(0,1)} + \ldots, \tag{4.1}$$

where $f^{(i,j)}$ is a function of r, z and the other parameters of the system. All unknown functions $(\mathbf{u}, P, c)$ are expanded in the form (4.1), as well as the given pressure and pressure gradient at $z = 0$ (P_I and G). Substitution of these expansions into the basic system will give sets of equations and boundary conditions corresponding to the coefficients of each $\epsilon^i \delta^j$.

One possible mathematical difficulty arises because ϵ is an expansion parameter multiplying highest-order z derivatives, and the general perturbation problem is singular. Specifically, since ϵ is small, second derivatives with respect to z are neglected in the first approximation, and the solution for the velocity field in the form (4.1) is not uniformly valid for all z, since it is unbounded as $|z| \to \infty$ (i.e. $\lim_{|z| \to \infty} |\mathbf{u}^{(0,1)}| \to \infty$). Physically, this means that the condition of small radial flux is violated at infinity.

5. Hydrodynamic solution

The solution of the hydrodynamic equations correct to order ϵ and δ is straightforward. The zeroth-order equations are the usual ones for flow through rigid infinite tubes. The inertial term $R\mathbf{u}^{(0,0)}.\nabla\omega^{(0,0)}$ in the $O(\epsilon)$ system disappears, since $u^{(0,0)} = w^{(0,0)}_{,z} = 0$, and the Reynolds number does not enter the solution to this order. The velocity and pressure field are of the form

$$u = \delta u^{(0,1)} + O(\epsilon^2, \delta^2, \epsilon\delta), \tag{5.1}$$

$$w = w^{(0,0)} + \epsilon w^{(1,0)} + \delta w^{(0,1)} + O(\ldots), \tag{5.2}$$

$$P = P^{(0,0)} + \epsilon P^{(1,0)} + \delta P^{(0,1)} + O(\ldots), \tag{5.3}$$

where

$$u^{(0,1)} = -4\{G^{(0,0)}z + P_I^{(0,0)} - P_0\}(\tfrac{1}{4}r^3 - \tfrac{1}{2}r), \tag{5.4}$$

$$w^{(0,0)} = \tfrac{1}{4}G^{(0,0)}(r^2 - 1), \tag{5.5}$$

$$w^{(1,0)} = \tfrac{1}{4}G^{(1,0)}(r^2 - 1), \tag{5.6}$$

$$w^{(0,1)} = 4\{\tfrac{1}{2}G^{(0,0)}z^2 + (P_I^{(0,0)} - P_0)z + G^{(0,1)}\}(r^2 - 1), \tag{5.7}$$

$$P^{(0,0)} = G^{(0,0)}z + P_I^{(0,0)}, \tag{5.8}$$

$$P^{(1,0)} = G^{(1,0)}z + P_I^{(1,0)}, \tag{5.9}$$

$$P^{(0,1)} = 16\{\tfrac{1}{6}G^{(0,0)}z^3 + \tfrac{1}{2}(P_I^{(0,0)} - P_0)z^2 + G^{(0,1)}z + P_I^{(0,1)}\}, \tag{5.10}$$

and

$$P = P_I^{(0,0)} + \epsilon P_I^{(1,0)} + \delta P_I^{(0,1)} + \ldots \quad \text{at} \quad z = 0, \tag{5.11}$$

$$P_{,z} = G^{(0,0)} + \epsilon G^{(1,0)} + \delta G^{(0,1)} + \ldots \quad \text{at} \quad z = 0 \tag{5.12}$$

are given constants.

The basic flow (zeroth-order) is Poiseuille flow through a tube, and the $w^{(1,0)}$ and $w^{(0,1)}$ terms are paraboloids of revolution in r, but only the δ term is axially dependent (5.7). In addition, the only non-zero component of radial velocity is due to ultra-filtration (δ term), and it is a linearly decreasing function of z with slope $G^{(0,0)}$ at the wall (5.4). The pressure gradient $P_{,z}$ is constant up to $O(\epsilon)$, with ultra-filtration causing a second-order variation in z (5.10). Clearly,

$$\lim_{|z|\to\infty} |\mathbf{u}^{(0,1)}| \to \infty \quad \text{and} \quad \lim_{|z|\to\infty} |P^{(0,1)}_{,z}| \to \infty,$$

so that this asymptotic form of solution breaks down for large enough z, as discussed in § 4. In the region $[0, L]$, however, the solution is a valid representation of a basic Poiseuille flow with a small radial convection (ultra-filtration) effect superimposed upon it, as well as a small ϵ effect.

Even though the aforementioned difficulties arise for large $|z|$, the expansion is valid in the finite region of practical interest $0 \leqslant z \leqslant L$. Therefore, the singular aspects of the problem need not be considered in this analysis, since we confine applicability of the solution to a bounded region of the z axis.

6. Mass-transport equations

The concentration is expressed as in (4.1):

$$c = c^{(0,0)} + \epsilon c^{(1,0)} + \delta c^{(0,1)} + \ldots, \tag{6.1}$$

so that there are three systems of equations corresponding to $c^{(0,0)}$, $c^{(1,0)}$ and $c^{(0,1)}$. These may be put in the general form

$$L[c^{(i,j)}] = F^{(i,j)}, \tag{6.2}$$

$$c^{(i,j)}_{,r} + \beta^{(0,0)} c^{(i,j)} = B^{(i,j)} \quad \text{on} \quad r = 1, \tag{6.3}$$

$$c^{(i,j)} = C^{(i,j)} \quad \text{at} \quad z = 0, \tag{6.4}$$

$$\lim_{r \to 0} |c^{(i,j)}| < \infty, \tag{6.5}$$

where

$$L[c] \equiv \frac{1}{r}(rc_{,r})_{,r} - w^{(0,0)} c_{,z}, \tag{6.6}$$

and $F^{(i,j)}_{(r,z)}$, $B^{(i,j)}_{(z)}$ and $C^{(i,j)}$ are inhomogeneous terms known for each (i,j) in terms of the lower-order solution

$$F^{(0,0)} = B^{(0,0)} = 0, \quad C^{(0,0)} = c_I, \tag{6.7}$$

$$F^{(1,0)} = w^{(1,0)} c^{(0,0)}_{,z}, \quad B^{(1,0)} = C^{(1,0)} = 0, \tag{6.8}$$

$$\left.\begin{aligned} F^{(0,1)} &= u^{(0,1)} c^{(0,0)}_{,r} + w^{(0,1)} c^{(0,0)}_{,z}, \\ B^{(0,1)} &= -\beta^{(0,1)} c^{(0,0)}, \quad C^{(0,1)} = 0. \end{aligned}\right\} \tag{6.9}$$

In addition, from (3.7) and (5.1),

$$\beta^{(0,0)} = S \quad \text{(constant Sherwood number)}, \tag{6.10}$$

$$\beta^{(0,1)} = (\hat{k} - 1)\, u^{(0,1)} \quad \text{at} \quad r = 1. \tag{6.11}$$

To determine c, we must first solve the (0, 0) system. From this result and the known hydrodynamic solution, we can then calculate $F^{(i,j)}$ and $B^{(i,j)}$ for the (1, 0) and (0, 1) systems, so that the inhomogeneous terms in (6.2) and (6.3) are known once $c^{(0,0)}$ is known. We can then proceed to solve for $c^{(1,0)}$ and $c^{(0,1)}$.

7. Zeroth-order solution

Since ultra-filtration is neglected to this order, the system is a variant of the classic Graetz problem in heat transfer (Kays 1966). However, as mentioned in § 1, the boundary condition is different in form, and comparison of the two solutions for corresponding constant Sherwood number S and local Nusselt number may be misleading.

The differential equation (6.2) and boundary condition (6.3) are homogeneous in this case, and a solution of separable form exists:

$$c^{(0,0)}(r,z) = R^{(0,0)}(r)\, Z^{(0,0)}(z), \tag{7.1}$$

where

$$Z^{(0,0)} = \exp\{-\lambda z\}, \tag{7.2}$$

and $R^{(0,0)}$ satisfies the second-order linear differential equation and boundary conditions

$$R^{(0,0)''} + \frac{1}{r} R^{(0,0)'} + \lambda w^{(0,0)} R^{(0,0)} = 0, \tag{7.3}$$

$$R^{(0,0)'} + S R^{(0,0)} = 0 \quad \text{at} \quad r = 1, \tag{7.4}$$

$$\lim_{r \to 0} |R^{(0,0)}| < \infty, \tag{7.5}$$

where

$$' \equiv d/dr.$$

Equations (7.3)–(7.5) define a Sturm–Liouville problem giving a set of eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{R_n^{(0,0)}\}$ for which the solution is non-trivial. Since the operator $r^{-1}(rR^{(0,0)\prime})'$ with boundary condition (7.4) is self-adjoint, the eigenvalues are real, and the eigenfunctions are orthogonal with respect to $rw^{(0,0)}$. Furthermore, the $\{R_n^{(0,0)}\}$ form a complete set, so that any function of r satisfying the boundary condition can be expanded in a series of them. The solution follows the usual method for eigenvalue problems of this type, giving

$$c^{(0,0)} = \sum_{n=1,2,3,\ldots}^{\infty} A_n \bar{R}_n^{(0,0)} \exp\{-\lambda_n z\}, \tag{7.6}$$

$$\bar{R}_n^{(0,0)} = \frac{R_n^{(0,0)}}{a_n}, \quad R_n^{(0,0)} = \sum_{m=0,2,4,\ldots}^{\infty} f_m^{(0,0)}(\lambda_n) r^m,$$

$$a_n^2 = \int_0^1 rw^{(0,0)}(R_n^{(0,0)})^2\, dr, \quad A_n = c_I \int_0^1 rw^{(0,0)} \bar{R}_n^{(0,0)}\, dr,$$

$$f_0^{(0,0)} = 1, \quad f_2^{(0,0)} = \tfrac{1}{4}\zeta, \quad f_{m+2}^{(0,0)} = \frac{\zeta}{(m+2)^2}\{f_m^{(0,0)} - f_{m-2}^{(0,0)}\} \quad \text{for} \quad m = 2,4,6,\ldots,$$

$$\zeta \equiv \tfrac{1}{4}\lambda G^{(0,0)}, \quad \sum_{m=0,2,4,\ldots}^{\infty} (m+S) f_m^{(0,0)} = 0.$$

From the form of the solution, we should note that $c^{(0,0)}$ is an even function of r, and that $c_{,r}^{(0,0)} = 0$ on $r = 0$ as a consequence. Furthermore, $c^{(0,0)}$ depends on dimensionless parameters S and $G^{(0,0)}$. However, the latter is a fixed number if W is defined as an average velocity, so it is not to be regarded as varying. The dependence of the solution on the actual dimensionless pressure gradient, or equivalently on W, enters through the dimensionless axial length.

8. Solution of ϵ and δ systems

8.1. *Transformed equations*

The partial differential equation and boundary conditions (6.2)–(6.5) can be reduced to a system of ordinary differential equations in r by taking their Laplace transform with respect to z. This approach is useful in these two cases, because $F^{(i,j)}$ and $B^{(i,j)}$ are expressed as infinite sums, and the equations are not separable, as is the (0, 0) problem. If we define the Laplace transforms as

$$\xi^{(i,j)} = \xi_{(r,s)}^{(i,j)} = \mathscr{L}\{c^{(i,j)}\}, \tag{8.1}$$

$$\hat{F}^{(i,j)} = \hat{F}_{(r,s)}^{(i,j)} = \mathscr{L}\{F^{(i,j)}\}, \tag{8.2}$$

$$\hat{B}^{(i,j)} = \hat{B}_{(s)}^{(i,j)} = \mathscr{L}\{B^{(i,j)}\}, \tag{8.3}$$

$$\Phi_k = \Phi_k(\lambda_n, s) = \mathscr{L}\{z^k \exp\{-\lambda_n z\}\}, \tag{8.4}$$

where

$$\mathscr{L}\{f\} \equiv \int_0^\infty f(r,z)\exp\{-sz\}\, dz, \quad \operatorname{Re} s > 0, \tag{8.5}$$

then the transformed equations are

$$\xi_{,rr}^{(i,j)} + \frac{1}{r}\xi_{,r}^{(i,j)} - sw^{(0,0)}\xi^{(i,j)} = \hat{F}^{(i,j)}, \tag{8.6}$$

$$\xi_{,r}^{(i,j)} + \beta^{(0,0)}\xi_{,r}^{(i,j)} = \hat{B}^{(i,j)} \quad \text{on} \quad r = 1, \tag{8.7}$$

$$\lim_{r\to 0} |\xi^{(i,j)}| < \infty. \tag{8.8}$$

Equations (8.6)–(8.8) are ordinary differential equations in r where $\xi^{(i,j)}$ is an analytic complex-valued function of s.

Since (8.6)–(8.8) are linear, we can divide the solution into a 'particular' $p^{(i,j)}$ and 'complementary' $q^{(i,j)}$ part defined below.

$$\xi^{(i,j)} = p^{(i,j)} + q^{(i,j)}, \tag{8.9}$$

where

$$p^{(i,j)}_{,rr} + \frac{1}{r} p^{(i,j)}_{,r} - sw^{(0,0)} p^{(i,j)} = \hat{F}^{(i,j)}, \tag{8.10}$$

$$\lim_{r \to 0} |p^{(i,j)}| < \infty, \tag{8.11}$$

and

$$q^{(i,j)}_{,rr} + \frac{1}{r} q^{(i,j)}_{,r} - sw^{(0,0)} q^{(i,j)} = 0, \tag{8.12}$$

$$q^{(i,j)}_{,r} + \beta^{(0,0)} q^{(i,j)} = \hat{B}^{(i,j)} - \hat{D}^{(i,j)} \quad \text{on} \quad r = 1, \tag{8.13}$$

$$\lim_{r \to 0} |q^{(i,j)}| < \infty, \tag{8.14}$$

where

$$\hat{D}^{(i,j)} \equiv p^{(i,j)}_{,r} + \beta^{(0,0)} p^{(i,j)}. \tag{8.15}$$

We first solve for $p^{(i,j)}$, calculate $\hat{D}^{(i,j)}$ from (8.15), then solve (8.12)–(8.14), which consists of a homogeneous equation with an inhomogeneous boundary condition. Further subdivision of the solution will be made for components of $\hat{F}^{(i,j)}$ and $\hat{B}^{(i,j)}$. Since these result from products of the $(0,0)$ solution and velocity, and since $c^{(0,0)}$ is expressed in the form of an infinite sum, it can be shown that $F^{(i,j)}$ and $B^{(i,j)}$ are sums over a finite number k of functions of the form

$$F^{(i,j)} = \sum_k F_k^{(i,j)}, \tag{8.16}$$

$$B^{(i,j)} = \sum_k B_k^{(i,j)}, \tag{8.17}$$

$$F_k^{(i,j)} = \sum_{n=1}^{\infty} f_{nk}^{(i,j)}(r)\, z^k \exp\{-\lambda_n z\}, \tag{8.18}$$

$$B_k^{(i,j)} = \sum_{n=1}^{\infty} b_{nk}^{(i,j)} z^k \exp\{-\lambda_n z\}; \tag{8.19}$$

thus it is useful to express the transformed concentration as

$$\xi^{(i,j)} = \sum_k \xi_k^{(i,j)}. \tag{8.20}$$

Assuming that integration and infinite summation can be interchanged, the transforms of (8.16)–(8.19) can be written as

$$\hat{F}^{(i,j)} = \sum_k \hat{F}^{(i,j)}, \tag{8.21}$$

$$\hat{B}^{(i,j)} = \sum_k \hat{B}^{(i,j)}, \tag{8.22}$$

$$\hat{F}_k^{(i,j)} = \sum_{n=1}^{\infty} f_{nk}^{(i,j)}(r)\, \Phi_k(\lambda_n, s), \tag{8.23}$$

$$\hat{B}_k^{(i,j)} = \sum_{n=1}^{\infty} b_{nk}^{(i,j)} \Phi_k(\lambda_n, s). \tag{8.24}$$

Furthermore, $f_{nk}^{(i,j)}$ can be expressed as a power series in r, because it arises from powers of r and the eigenfunctions $\bar{R}_n^{(0,0)}$ and $\bar{R}_n^{(0,0)\prime}$:

$$f_{nk}^{(i,j)} = \sum_{m=0,2,4,\ldots}^{\infty} a_{mk}^{(i,j)}(\lambda_n)\, r^m. \tag{8.25}$$

Putting (8.25) into (8.23) allows us to rewrite $\hat{F}_k^{(i,j)}$:

$$\hat{F}_k^{(i,j)} = \sum_{m=0,2,4,\ldots}^{\infty} g_{mk}^{(i,j)}(s)\, r^m, \tag{8.26}$$

where

$$g_{mk}^{(i,j)} = \sum_{n=1,2,3,\ldots}^{\infty} a_{mk}^{(i,j)}(\lambda_n)\, \Phi_k(\lambda_n, s). \tag{8.27}$$

The $a_{mk}^{(i,j)}$ and $b_{nk}^{(i,j)}$ are known from the $(0,0)$ solution alone. Specifically, for the ϵ case, we can show that there is only one term ($k = 0$) and that

$$a_{00}^{(1,0)}(\lambda_n) = x_{n0}^{(1,0)}, \quad a_{m0}^{(1,0)} = x_{n0}^{(1,0)}(f_m^{(0,0)} - f_{m-2}^{(0,0)}), \quad m = 2, 4, 6, \ldots,$$

where $x_{n0}^{(1,0)} \equiv \lambda_n A_n G^{(1,0)}/4a_n$, and lastly that $b_{n0}^{(1,0)} = 0$. In the same way, $a_{mk}^{(0,1)}$ and $b_{nk}^{(0,1)}$ can be calculated, although there are three terms ($k = 0, 1, 2$) and the algebra is more complicated in that case.

The systems of transformed equations can now be listed as

$$\xi^{(i,j)} = \sum_k \xi_k^{(i,j)}, \tag{8.28}$$

$$\xi_k^{(i,j)} = p_k^{(i,j)} + q_k^{(i,j)}. \tag{8.29}$$

(i) Particular:

$$p_{k,rr}^{(i,j)} + \frac{1}{r} p_{k,r}^{(i,j)} - s w^{(0,0)} p_k^{(i,j)} = \sum_{m=0,2,4,\ldots}^{\infty} g_{mk}^{(i,j)}(s)\, r^m, \tag{8.30}$$

$$\lim_{r\to 0} |p_k^{(i,j)}| < \infty. \tag{8.31}$$

(ii) Complementary:

$$q_{k,rr}^{(i,j)} + \frac{1}{r} q_{k,r}^{(i,j)} - s w^{(0,0)} q_k^{(i,j)} = 0, \tag{8.32}$$

$$q_{k,r}^{(i,j)} + \beta^{(0,0)} q_k^{(i,j)} = \hat{B}_k^{(i,j)} - \hat{D}_k^{(i,j)} \quad \text{on} \quad r = 1, \tag{8.33}$$

$$\lim_{r\to 0} |q_k^{(i,j)}| < \infty, \tag{8.34}$$

$$\hat{D}_k^{(i,j)} \equiv p_{k,r}^{(i,j)} + \beta^{(0,0)} p_k^{(i,j)}, \tag{8.35}$$

where

$$\hat{B}_k^{(i,j)} = \sum_{n=1,2,\ldots}^{\infty} b_{nk}^{(i,j)} \Phi_k, \tag{8.36}$$

$$g_{mk}^{(i,j)} = \sum_{n=1,2,\ldots}^{\infty} a_{mk}^{(i,j)}(\lambda_n)\, \Phi_k. \tag{8.37}$$

The method of solution involves, first, the calculation of $a_{mk}^{(i,j)}$, $b_{nk}^{(i,j)}$ and Φ_k for each k using the $(0,0)$ solution, which has previously been found. Next, we obtain $g_{mk}^{(i,j)}$ and find a particular solution using a power series in r. We then calculate $\hat{D}_k^{(i,j)}$ and $\hat{B}_k^{(i,j)}$, solve (8.32)–(8.34), add $p_k^{(i,j)}$ and $q_k^{(i,j)}$, and sum over k. The details involved will be considered in §§ 8.2 and 8.3.

8.2. *Particular solution*

A power series solution of (8.30) in which boundary conditions are not specified can be found by equating coefficients of r^m, giving the result

$$p_k^{(i,j)} = \sum_{m=0,2,4,\ldots}^{\infty} p_{mk}^{(i,j)}(s)\, r^{m+2}, \tag{8.38}$$

where

$$p_{0k}^{(i,j)} = \tfrac{1}{4} g_{0k}^{(i,j)}, \tag{8.39}$$

$$16 p_{2k}^{(i,j)} - \theta p_{0k}^{(i,j)} = g_{2k}^{(i,j)},$$

$$p_{m+2,k}^{(i,j)} = (m+2)^{-2}\{g_{m+2,k}^{(i,j)} + \theta(p_{mk}^{(i,j)} - p_{m-2,k}^{(i,j)})\} \quad \text{for} \quad m = 2, 4, 6, \ldots,$$

$$\theta \equiv -\tfrac{1}{4} s G^{(0,0)}.$$

Equations (8.38)–(8.39) give a particular transformed solution for each set of $\{g_{mk}^{(i,j)}\}$ derived from the (0, 0) solution.

8.3. *Complementary solution*

Calculation of $\hat{D}_k^{(i,j)}$ from (8.35) yields

$$\hat{D}_k^{(i,j)} = \sum_{m=2,4,6,\ldots}^{\infty} (m + \beta^{(0,0)})\, p_{mk}^{(i,j)}. \tag{8.40}$$

Since $\hat{B}_k^{(i,j)}$ can be obtained from results of the (0, 0) solution, the sum $\hat{B}_k^{(i,j)} - \hat{D}_k^{(i,j)}$ in (8.33) is known. Equations (8.32)–(8.34) are similar to (7.3)–(7.5), with the exception of an inhomogeneous boundary condition in this case. Accordingly, we can find a series solution:

$$q_k^{(i,j)} = \hat{A}_k^{(i,j)}(s) \sum_{m=0,2,4,\ldots}^{\infty} \bar{q}_m(s)\, r^m, \tag{8.41}$$

where the coefficients $\{\bar{q}_m\}$ are determined by

$$\left.\begin{aligned} &\bar{q}_0 = 1, \quad \bar{q}_2 = \tfrac{1}{2}\theta, \\ &\bar{q}_{m+2} = \frac{\theta}{(m+2)^2}\{\bar{q}_m - \bar{q}_{m-2}\} \quad \text{for} \quad m = 2, 4, 6 \ldots, \end{aligned}\right\} \tag{8.42}$$

and

$$\hat{A}_k^{(i,j)} = \frac{\hat{B}_k^{(i,j)} - \hat{D}_k^{(i,j)}}{\hat{S}(s)}, \tag{8.43}$$

$$\hat{S}(s) = \sum_{m=0,2,4,\ldots}^{\infty} (m + \beta^{(0,0)})\, \bar{q}_m. \tag{8.44}$$

8.4. *Total solution*

The concentration is obtained by inverting algebraically the transformed total concentration $\xi^{(i,j)}$ (8.28):

$$c^{(i,j)}(r,z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \xi^{(i,j)}(r,s) \exp\{sz\}\, ds. \tag{8.45}$$

From the resulting equations, we can immediately observe that $\xi^{(i,j)}$ is singular at those points on the negative real s axis corresponding to the eigenvalues $s = -\lambda_n$. This occurs in the particular solution since Φ_k and consequently $g_{mk}^{(i,j)}$

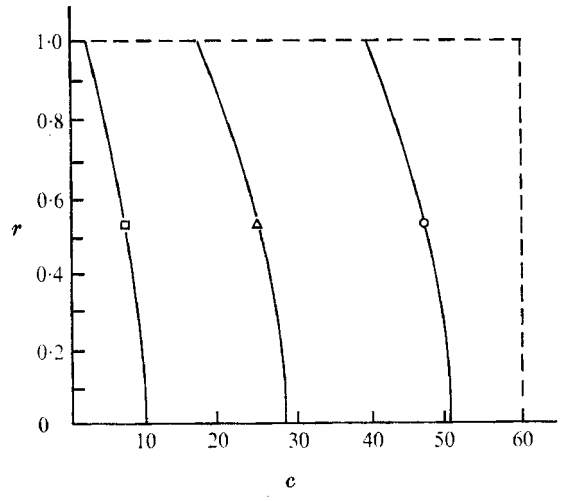

FIGURE 2. Total concentration profiles at $z = L$ for varying values of tube diameter holding the volume flow rate fixed. r, c dimensionless. Diameter (cm): ○, 0·01; △, 0·02; □, 0·04.

have poles there, causing $p_k^{(i,j)}$ to be singular. In the complementary case, $\hat{A}_k^{(i,j)}$ is singular at $s = -\lambda_n$, because these points are roots of the eigenvalue equation (i.e. $\hat{S}(s) = 0$ at $s = -\lambda_n$).

9. Discussion

Results from the model were obtained for the case of primary interest in hemodialysis: the effect of ultra-filtration on the concentration profile. To this end, the pressure and pressure gradient at $z = 0$ were taken arbitrarily to be given numbers independent of both ϵ and δ (i.e. $P_{,z}|_{z=0} = G^{(0,0)}$, $P|_{z=0} = P_I^{(0,0)}$), so that the contribution of $c^{(1,0)}$ to the total solution was zero and $c = c^{(0,0)} + \delta c^{(0,1)}$ to first order. The result that $c^{(1,0)} = 0$ when $G^{(1,0)} = 0$ could have been deduced directly from the equations in § 6; however, the ϵ terms were retained at that point since the mathematics involved in both the δ and ϵ solutions was the same (§ 8).

Figures 2, 4 and 5 show the total concentration $c^{(0,0)} + \delta c^{(0,1)}$ as a function of r at distance L down the tube, as three parameters vary holding the others fixed at their basic values. Since $c^{(0,0)}$ is an order of magnitude greater than $\delta c^{(0,1)}$, subtle changes in the higher-order concentration term are obscured in figures 2–5, which necessitates considering $\delta c^{(0,1)}$ alone (figures 6–9). The entrance concentration is taken arbitrarily to be $c_I = 100$ in all the figures, and the basic values of the parameters are $D = 1{\cdot}8(10^{-5})\,\mathrm{cm^2\,s^{-1}}$, $\hat{P} = 1{\cdot}1(10^{-3})\,\mathrm{cm\,s^{-1}}$, $\hat{k} = 0{\cdot}9$, $a = 0{\cdot}01$ cm, $L = 7{\cdot}62$ cm, $W = 1\,\mathrm{cm\,s^{-1}}$, $P_{\mathrm{ult}} = 100$ mm Hg, and $K = 10^{-11}$ cm. The values of diffusivity, permeability and transmittance correspond to the solute urea.

The effect of changing the tube radius while holding the flow rate fixed, as well as the other parameters, is given in figure 2. As the radius a increases, the mean axial velocity decreases as a^{-2}, which allows more time for diffusion to occur as the particle travels distance L (i.e. 'residence time' increases). This should cause

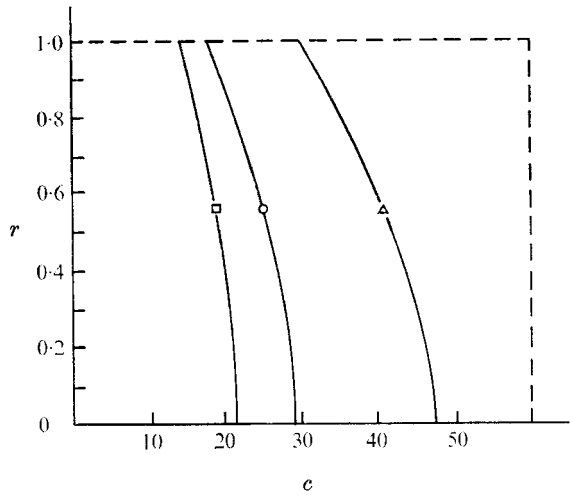

FIGURE 3. Total concentration profiles at three distances down the tube. r, c dimensionless. L (cm): ○, 7·62; △, 5·08; □, 10·16.

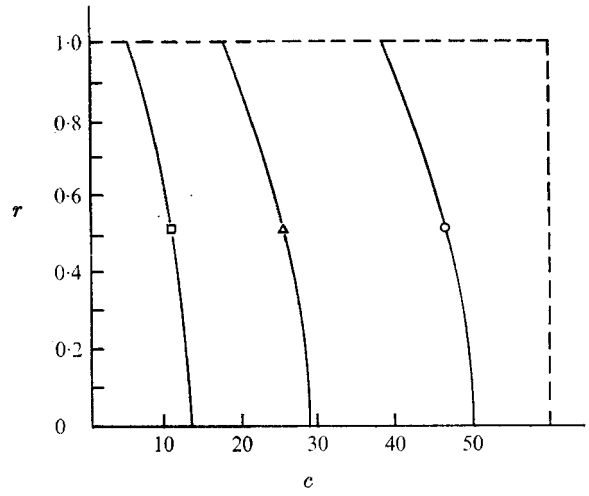

FIGURE 4. Total concentration profiles at $z = L$ for varying membrane permeability. r, c dimensionless. $\hat{P}$ (cm min^{-1}): ○, 0·0335; △, 0·067; □, 0·134.

the concentration at L to decrease. However, there is an opposing effect, since the surface area to volume ratio decreases as a^{-1} as a increases. The large decreases in concentration in figure 2 with doubling values of a clearly indicate that the residence time of a particle is dominant over the area/volume ratio to a significant extent.

The behaviour of the solution at three distances down the tube is shown in figure 3. The resulting curves are typical for solutions that can be represented as a sum of decaying exponentials in z, as is the case here. In the actual computation, five eigenvalues were determined giving values ranging from $O(1)$ to $O(150)$, so that for all z greater than some short distance down the tube, all except the first

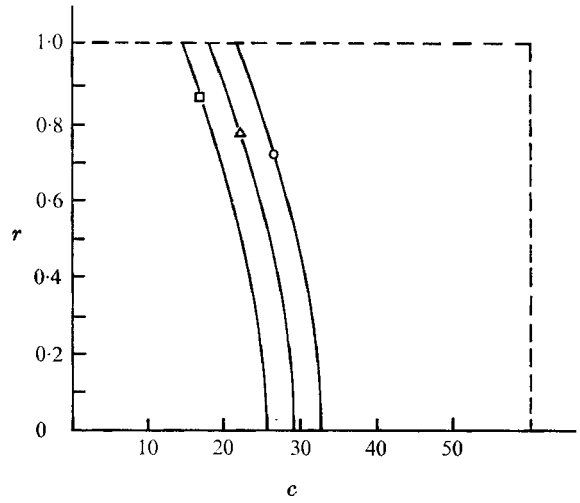

FIGURE 5. Total concentration profiles at $z = L$ for varying values of ultra-filtration pressure $P|_{z=L} - P_0$. r, c dimensionless. P_{ult} (mm Hg): ○, 0; △, 100; □, 200.

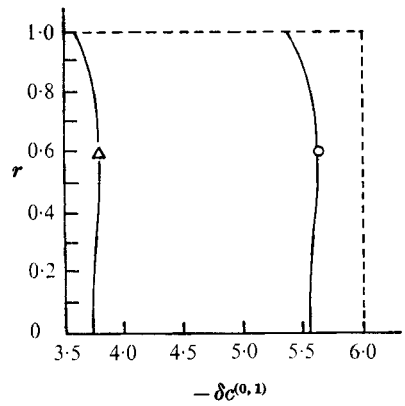

FIGURE 6. Higher-order concentration profiles $\delta c^{(0,1)}$ at $z = 7{\cdot}62$ cm for varying values of ultra-filtration pressure. r, $\delta c^{(0,1)}$ dimensionless. P_{ult} (mm Hg): △, 100; ○, 150.

few exponential terms are negligible. Obviously, though, as $z \to 0$, all the exponential terms are approaching $O(1)$ and more λ_n's are needed to approximate the exact solution well.

The membrane permeability influences the total solution via boundary condition (3.7), by changing values of Sherwood number S, which can be considered the ratio of the speed of diffusive transport through the membrane to the speed of radial diffusion. Since the computed results show that each of the eigenvalues are monotone increasing functions of S, as the permeability increases, so will the eigenvalues thereby decreasing c at $z = L$, as seen in figure 4 (also see the appendix). Note that the curves in figure 4 are close in value to those in figure 2, since the zeroth-order solution is a function of S, so that the effects of varying radius and permeability appear only at higher order.

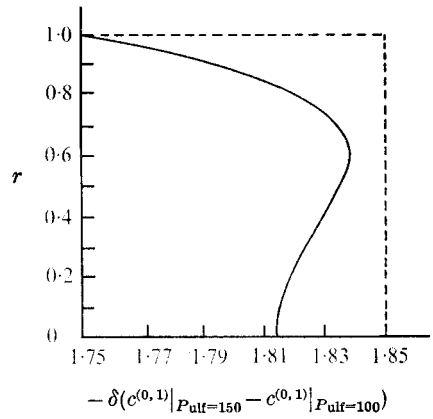

FIGURE 7. Higher-order concentration difference profile $\Delta\delta c^{(0,1)}$ at $z = L$ for ultra-filtration pressures of 100 and 150 mmHg. r, $-\delta(c^{(0,1)}|_{P_{ult}=150} - c^{(0,1)}|_{P_{ult}=100})$ dimensionless.

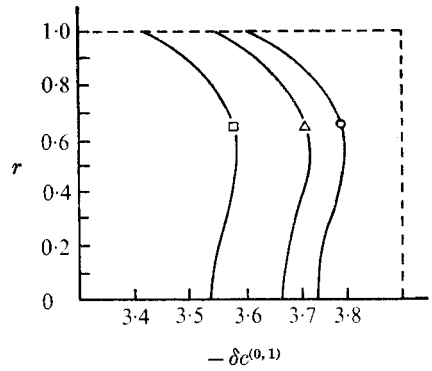

FIGURE 8. Higher-order concentration profiles $\delta c^{(0,1)}$ at $z = L$ for varying membrane transmittance. r, $-\delta c^{(0,1)}$ dimensionless. $\hat{k}$: ○, 0·9; △, 0·7; □, 0·3.

To examine the effect of ultra-filtration on the solution, it is necessary to consider the higher-order term $\delta c^{(0,1)}$ separately, since only the gross results are demonstrated in the total concentration curves of figure 5. Note that $c^{(0,1)}$ is negative in figures 6–9. Since variations in the hydrodynamic permeability K cause only uninteresting linear changes in δ, we choose to look at the effects of varying the 'ultra-filtration pressure' (defined in this paper as the pressure difference between the blood at distance L and the dialysate) and the transmittance $\hat{k}$ of the chemical solute through the membrane. Recall that $0 \leqslant \hat{k} \leqslant 1$ and $\hat{k} \to 1$ as the molecular size of the chemical solute becomes much smaller than some imagined membrane pore size, and we approach zero retardation. From boundary condition equation (3.5), it might seem at first glance that the effects of both $\hat{k}$ and ultra-filtration pressure changes should be the same, since (5.4) shows that the radial velocity is directly proportional to dialysate pressure P_0 and that $\hat{k}$ multiplies u in (3.5). However, u also enters in the convective acceleration term in (2.7) and comparison of the magnitudes of $\delta c^{(0,1)}$ in figures 6 and 7 with figures 8 and 9 clearly shows that changes in ultra-filtration pressure are much more significant than the same order of magnitude changes in transmittance.

The fact that the concentration at $z = L$ decreases as the transmittance increases can be explained physically by the increase in the mass flux of solute convected through the membrane. However, the result that increased ultra-filtration also causes a decrease in concentration for any $\hat{k}$ seems to be contrary to the expectation that the concentration would increase, since more solute per unit volume is retained inside the tube than is convected through it. Physically, the explanation lies in the fact that as ultra-filtration pressure increases, more fluid is drawn out through the membrane so that there is a corresponding drop in the axial velocity and the time a fluid particle takes in travelling a distance L increases, allowing more ordinary diffusion to take place. Since diffusion is more significant than ultra-filtration, the net result is a decrease in the concentration at $z = L$.

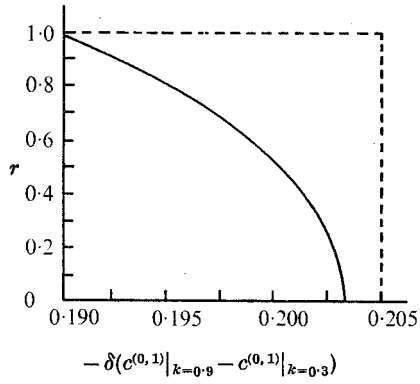

FIGURE 9. Higher-order concentration difference profiles at $z = L$ for transmittances of 0·9 and 0·3. r, $-\delta(c^{(0,1)}|_{k=0\cdot9} - c^{(0,1)}|_{k=0\cdot3})$ dimensionless.

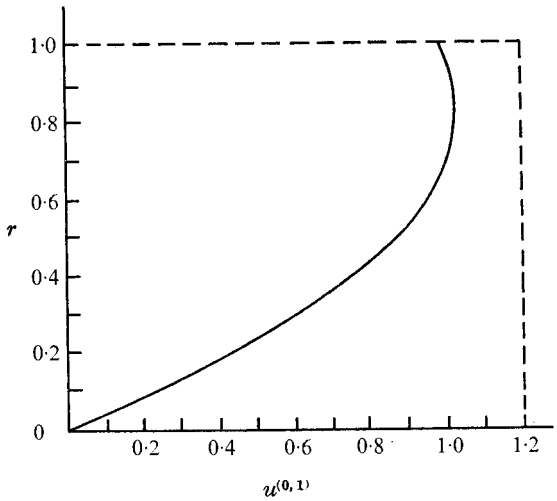

FIGURE 10. Higher-order radial velocity profile normalized so that $u^{(0,1)}$ at the membrane = 1. r, $u(0, 1)$ dimensionless.

A simplified model of the equations, presented in the appendix, indicates that the concentration decreases as both S and $\hat{k}$ increase and as either u or w decreases. However, the axial velocity w is dominant and it decreases by continuity as u increases, so that the net effect is a decrease in concentration as u increases. Of course, if the axial velocity were kept constant, this result would be lost and the concentration would increase as ultra-filtration increases, since there would be no opposing w effect.

The curves in figures 6 and 8 have basically the same profile shape when plotted on the same scale. It is noteworthy that the maximum negative values of $\delta c^{(0,1)}$ are roughly at $r \cong 0{\cdot}6$, and not at the membrane itself, as might have been expected. This may be due to the fact that the radial velocity (figure 10) has a maximum at $r \cong 0{\cdot}816$, while the zeroth-order concentration is maximum at the tube centre $r = 0$ so the maximum radial solute flux lies somewhere in

$0 < r < 0{\cdot}816$. A similar arguement can be used to explain the curve showing the difference profile for two values of ultra-filtration pressure (figure 7). In addition, the build-up in concentration near the membrane (concentration polarization) is evident from the figures; however, the hydrodynamics are not affected, as they are in reverse osmosis.

Quite different behaviour is shown in figure 9, which gives the maximum effect of the transmittance difference at $r = 0$, not at the membrane. This is surprising, since one would expect the major difference to be at $r = 1$, owing to a build-up in concentration at the membrane as $\hat{k}$ decreases. Deeper reflexion, though, indicates that there are possibly two physical effects to consider. As $\hat{k}$ decreases, with the ultra-filtration velocity fixed, there is an increase in concentration near the membrane, so that the total concentration $c(1, L)$ increases near the membrane. Since the dialysate concentration is taken to be zero, there is a larger difference in concentration across the membrane, causing more ordinary diffusion across it. Thus mass transfer due to $\hat{k}$ and ordinary diffusion are coupled in this sense, a decrease in the former causing an increase in the latter. Furthermore, as $c(1, L)$ increases, the difference in the concentration gradient within the blood will decrease (i.e. $c(0, L) - c(1, L)$ decreases), so that the radial diffusion within the tube is affected, possibly exerting its major influence at $r = 0$, as indicated in figure 9.

The author thanks Professor A. L. Babb of the Chemical Engineering Department at the University of Washington in Seatle for suggesting this problem, and Mr R. McKee for assistance with the numerical calculations. This work was supported by Contract PH 43-66-932 from the National Institute of Arthritis and Metabolic Diseases, NIH, DHEW.

Appendix

A simplified model of (2.7) and (3.6) has been developed, in order to examine the effect of changes in concentration due to some of the parameters, notably the radial velocity. Neglecting $uc_{,r}$ in (2.7), the equations are

$$wc_{,z} = c_{,rr},$$
$$c_{,r} + \beta c = 0 \quad \text{on} \quad r = 1,$$
$$\beta = (\hat{k} - 1)\, u + S.$$

Regarding u and w as constants, and requiring c to be an even function of r, we can solve easily by separation of variables, to obtain $c = \exp\{-\lambda z\} \cos{(\lambda w)^{\frac{1}{2}}} r$. The boundary condition at $r = 1$ gives the transcendental equation

$$-(\lambda w)^{\frac{1}{2}} \sin{(\lambda w)^{\frac{1}{2}}} + \beta \cos{(\lambda w)^{\frac{1}{2}}} = 0.$$

Solving for the (dominant) small eigenvalue,

$$\lambda = \beta/[w(1 + 0{\cdot}5\beta)],$$

which means that λ increases from 0 to $2/w$ as β increases from 0 to ∞, and that $d\lambda/d\beta$ is a monotone decreasing function of β.

Since the concentration at $z = L$ decreases as λ increases, it follows that this

will occur when (i) S increases, (ii) $\hat{k}$ increases if $S > u$, (iii) u decreases if $\hat{k} < 1$, (iv) w decreases. Therefore, from (iii) and (iv), decreasing both radial and axial velocity will cause a decrease in the concentration. However, these effects are not independent, since decreasing one will increase the other via continuity, so they actually oppose each other. From this result, it appears likely that a decrease in w due to increasing u is dominant over an increase in β due to decreasing u, so that the former effect will be more significant. Thus, increasing the ultra-filtration can be expected to decrease the concentration at $z = L$. This agrees with the findings obtained from the full set of equations.

REFERENCES

ARIS, R. 1962 *Vectors, Tensors and the Basic Equations of Fluid Mechanics.* Prentice-Hall.

BERMAN, A. 1953 Laminar flow in channels with porous walls. *J. Appl. Phys.* **24**, 1232–1235.

BIRD, R., STEWART, W. & LIGHTFOOT, E. 1960 *Transport Phenomena.* Wiley.

BRIAN, P. 1966 Mass transport in reverse osmosis. *Desalination by Reverse Osmosis* (ed. Merten), pp. 161–202. MIT Press.

COLTON, C., SMITH, K., STROEVE, P. & MERRILL, E. 1971 Laminar flow mass transfer in a flat duct with permeable walls. *Am. Inst. Chem. Eng.* **17**, 773–780.

DE BYE, J. & SCHENK, J. 1953 *Appl. Sci. Res.* A **3**, 308.

GOLDSTEIN, S. 1938 *Modern Developments in Fluid Dynamics,* vol. 2. Clarendon Press.

GRIMSRUD, L. 1965 Ph.D. thesis, University of Washington, Seattle.

GRIMSRUD, L. & BABB, A. L. 1966 Velocity and concentration profiles for laminar flow of a Newtonian fluid in a dialyzer. *Chem. Eng. Prog. Symposium Series,* **62**, 20–31.

KAYS, W. 1966 *Convective Heat and Mass Transfer.* McGraw-Hill.

LIU, M. 1971 Iterative analysis of a continuous system for desalination by reverse osmosis. *Desalination,* **9**, 181–191.

POPOVICH, R. P. 1971 Ph.D. thesis, University of Washington, Seattle.

POPOVICH, R., CHRISTOPHER, G. & BABB, A. L. 1971 The effects of membrane diffusion and ultra-filtration properties on hemodialyzer design and performance. *Chem. Eng. Prog. Symp.* **67**, 105–115.

PROBSTEIN, R. 1972 Desalination: some fluid mechanical problems. *Trans. A.S.M.E., J. Basic Eng.* D **94**, 286–313.